2. (a) $D = g_{xx}(0,2)g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D < 0$, g has a saddle point at $(0,2)$ by the Second Derivatives Test.

(b) $D = g_{xx}(0,2)g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D > 0$ and $g_{xx}(0,2) < 0$, g has a local maximum at $(0,2)$ by the Second Derivatives Test.

(c) $D = g_{xx}(0,2)g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about g at the point $(0,2)$.

4. In the figure, points at approximately $(-1, 1)$ and $(-1, -1)$ are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minima at or near $(-1, \pm 1)$. Similarly, the point $(1, 0)$ appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points $(-1, 0)$, $(1, 1)$, and $(1, -1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

To confirm our predictions, we have $f(x, y) = 3x - x^3 - 2y^2 + y^4 \Rightarrow f_x(x, y) = 3 - 3x^2$, $f_y(x, y) = -4y + 4y^3$. Setting these partial derivatives equal to 0, we have $3 - 3x^2 = 0 \Rightarrow x = \pm 1$ and $-4y + 4y^3 = 0 \Rightarrow y(y^2 - 1) = 0 \Rightarrow y = 0, \pm 1$. So our critical points are $(\pm 1, 0)$, $(\pm 1, \pm 1)$. The second partial derivatives are $f_{xx}(x, y) = -6x$, $f_{xy}(x, y) = 0$, and $f_{yy}(x, y) = 12y^2 - 4$, so $D(x, y) = f_{xx}(x, y) f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (-6x)(12y^2 - 4) - (0)^2 = -72xy^2 + 24x$. We use the Second Derivatives Test to classify the 6 critical points:

| Critical Point	D	f_{xx}	Conclusion
$(1, 0)$	24	-6	$D > 0$, $f_{xx} < 0 \Rightarrow f$ has a local maximum at $(1, 0)$
$(1, 1)$	-48		$D < 0 \Rightarrow f$ has a saddle point at $(1, 1)$
$(1, -1)$	-48		$D < 0 \Rightarrow f$ has a saddle point at $(1, -1)$
$(-1, 0)$	-24		$D < 0 \Rightarrow f$ has a saddle point at $(-1, 0)$
$(-1, 1)$	48	6	$D > 0$, $f_{xx} > 0 \Rightarrow f$ has a local minimum at $(-1, 1)$
$(-1, -1)$	48	6	$D > 0$, $f_{xx} > 0 \Rightarrow f$ has a local minimum at $(-1, -1)$

6. $f(x, y) = x^3y + 12x^2 - 8y \Rightarrow f_x = 3x^2y + 24x, f_y = x^3 - 8,$
$f_{xx} = 6xy + 24, f_{xy} = 3x^2, f_{yy} = 0.$ Then $f_y = 0$ implies $x = 2,$
and substitution into $f_x = 0$ gives $12y + 48 = 0 \Rightarrow y = -4.$
Thus, the only critical point is $(2, -4).$

$D(2, -4) = (-24)(0) - 12^2 = -144 < 0,$ so $(2, -4)$ is a saddle
point.
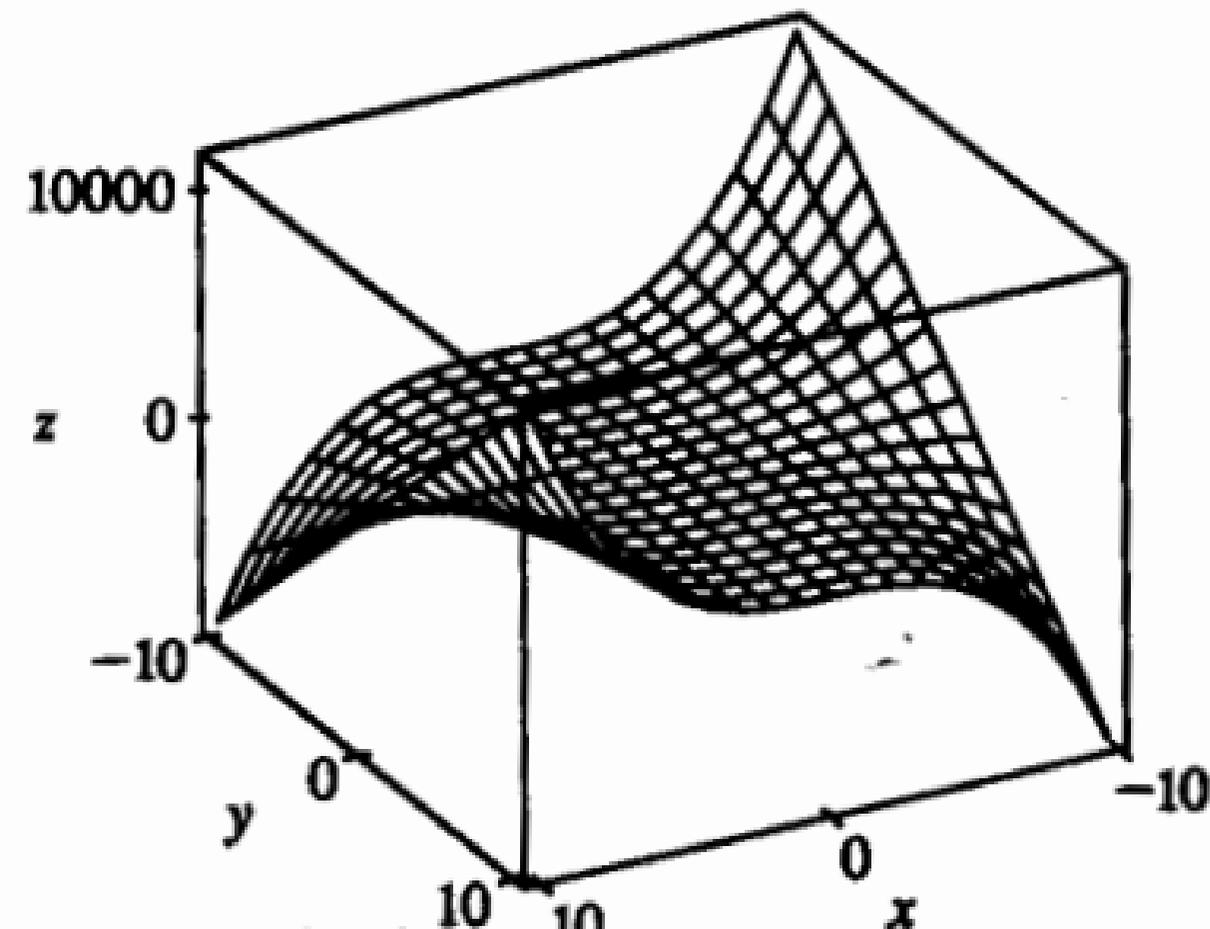
8. $f(x, y) = e^{4y - x^2 - y^2} \implies f_x = -2x e^{4y - x^2 - y^2}$,

$f_y = (4 - 2y)e^{4y - x^2 - y^2}$, $f_{xx} = (4x^2 - 2)e^{4y - x^2 - y^2}$,

$f_{xy} = -2x(4 - 2y)e^{4y - x^2 - y^2}$,

$f_{yy} = (4y^2 - 16y + 14)e^{4y - x^2 - y^2}$. Then $f_x = 0$ and $f_y = 0$

implies $x = 0$ and $y = 2$, so the only critical point is $(0, 2)$.

$D(0, 2) = (-2e^4)(-2e^4) - 0^2 = 4e^8 > 0$ and

$f_{xx}(0, 2) = -2e^4 < 0$, so $f(0, 2) = e^4$ is a local maximum.
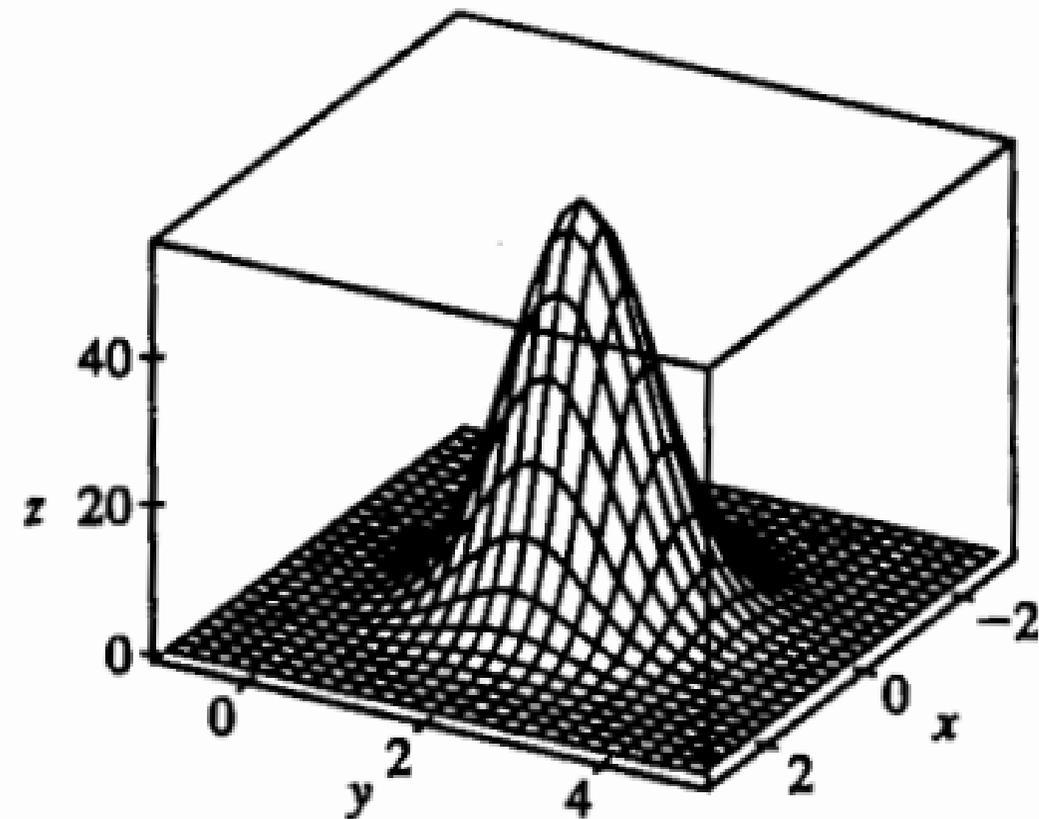
12. $f(x,y) = x^2 + y^2 + \dfrac{1}{x^2y^2} \quad \Rightarrow \quad f_x = 2x - 2x^{-3}y^{-2}$,

$f_y = 2y - 2x^{-2}y^{-3}$, $f_{xx} = 2 + 6x^{-4}y^{-2}$, $f_{yy} = 2 + 6x^{-2}y^{-4}$,

$f_{xy} = 4x^{-3}y^{-3}$. Then $f_x = 0$ implies $2x^4y^2 - 2 = 0$ or $x^4y^2 = 1$

or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $f_y = 0$
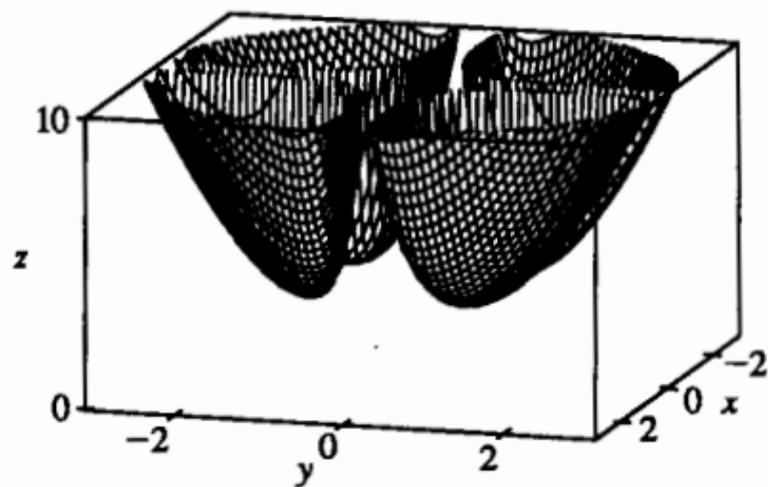
implies $2x^2y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies

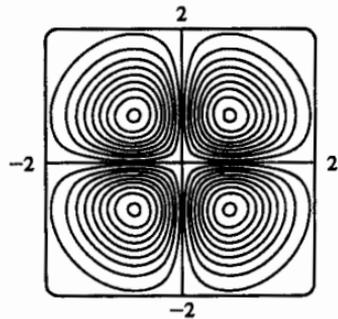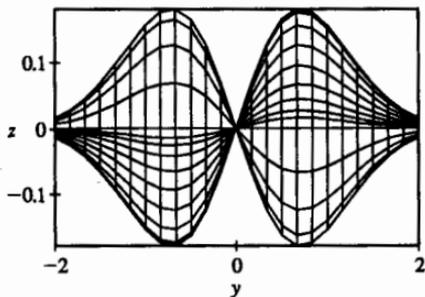$2x^{-6} - 2 = 0$ or $x^6 = 1$. Thus $x = \pm 1$ and if $x = 1$, $y = \pm 1$; if
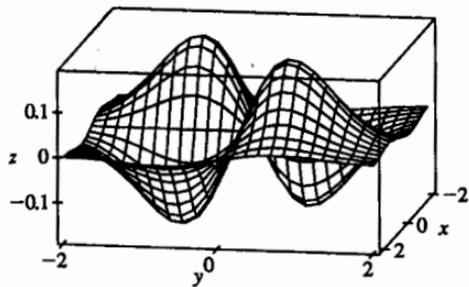
$x = -1$, $y = \pm 1$. So the critical points are $(1, 1)$, $(1, -1)$,

$(-1, 1)$ and $(-1, -1)$. Now $D(\pm 1, \pm 1) = D(\pm 1, \mp 1) = 64 - 16 > 0$ and $f_{xx} > 0$ always, so

$f(\pm 1, \pm 1) = f(\pm 1, \mp 1) = 3$ are local minima.

16. $f(x, y) = xye^{-x^2-y^2}$

There appear to be local maxima of about $f(\pm 0.7, \pm 0.7) \approx 0.18$ and local minima of about $f(\pm 0.7, \mp 0.7) \approx -0.18$. Also, there seems to be a saddle point at the origin.

$f_x = ye^{-x^2-y^2}(1-2x^2)$, $f_y = xe^{-x^2-y^2}(1-2y^2)$, $f_{xx} = 2xye^{-x^2-y^2}(2x^2-3)$,

$f_{yy} = 2xye^{-x^2-y^2}(2y^2-3)$, $f_{xy} = (1-2x^2)e^{-x^2-y^2}(1-2y^2)$. Then $f_x = 0$ implies $y = 0$ or $x = \pm\frac{1}{\sqrt{2}}$.

Substituting these values into $f_y = 0$ gives the critical points $(0,0)$, $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. Then

$D(x, y) = e^{2(-x^2-y^2)}\left[4x^2y^2(2x^2-3)(2y^2-3) - (1-2x^2)^2(1-2y^2)^2\right]$, so $D(0,0) = -1$, while

$D\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$ and $D\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$. But $f_{xx}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) < 0$, $f_{xx}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) > 0$,

$f_{xx}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) > 0$ and $f_{xx}\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) < 0$. Hence $(0,0)$ is a saddle point;

$f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2e}$ are local minima and $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2e}$ are local maxima.

26. $f_x(x, y) = 4 - 2x$ and $f_y(x, y) = 6 - 2y$, so the only critical point is $(2, 3)$ (which is in D) where $f(2, 3) = 13$.

Along L_1: $y = 0$, so $f(x, 0) = 4x - x^2 = -(x - 2)^2 + 4$, $0 \leq x \leq 4$, which has a maximum value when $x = 2$

where $f(2, 0) = 4$ and a minimum value both when $x = 0$ and $x = 4$, where $f(0, 0) = f(4, 0) = 0$. Along L_2:

$x = 4$, so $f(4, y) = 6y - y^2 = -(y - 3)^2 + 9$, $0 \leq y \leq 5$, which has a maximum value when $y = 3$ where

$f(4, 3) = 9$ and a minimum value when $y = 0$ where $f(4, 0) = 0$. Along L_3: $y = 5$, so

$f(x, 5) = -x^2 + 4x + 5 = -(x - 2)^2 + 9$, $0 \leq x \leq 4$, which has a maximum value when $x = 2$ where

$f(2, 5) = 9$ and a minimum value both when $x = 0$ and $x = 4$, where $f(0, 5) = f(4, 5) = 5$. Along L_4: $x = 0$, so

$f(0, y) = 6y - y^2 = -(y - 3)^2 + 9$, $0 \leq y \leq 5$, which has a maximum value when $y = 3$ where $f(0, 3) = 9$ and

a minimum value when $y = 0$ where $f(0, 0) = 0$. Thus the absolute maximum is $f(2, 3) = 13$ and the absolute

minimum is attained at both $(0, 0)$ and $(4, 0)$, where $f(0, 0) = f(4, 0) = 0$.
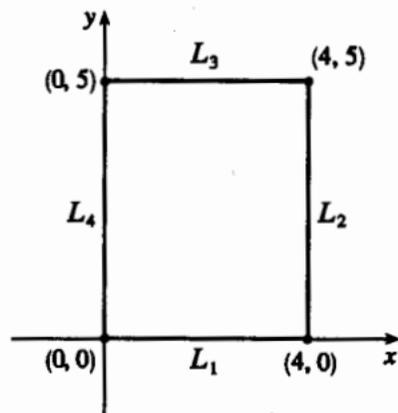
38. Here maximize $f(x, y) = xy \dfrac{(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2)^{1/2}}{a^2b^2}$. Then

$$f_x = yc^2 \dfrac{a^2b^2 - 2b^2x^2 - a^2y^2}{a^2b^2 (a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2)^{1/2}} \text{ and } f_y = xc^2 \dfrac{a^2b^2 - 2a^2y^2 - b^2x^2}{a^2b^2 (a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2)^{1/2}}. \text{ Then}$$

$f_x = 0$ (with $x, y > 0$) implies $y^2 = \dfrac{a^2b^2 - 2b^2x^2}{a^2}$ and substituting into $f_y = 0$ implies $3b^2x^2 = a^2b^2$ or

$x = \frac{1}{\sqrt{3}} a$, $y = \frac{1}{\sqrt{3}} b$ and then $z = \frac{1}{\sqrt{3}} c$. Thus the maximum volume of such a rectangle is

$V = (2x)(2y)(2z) = \frac{8}{3\sqrt{3}} abc$.

45. Note that here the variables are m and b, and $f(m,b) = \sum_{i=1}^{n} [y_i - (mx_i + b)]^2$. Then

$f_m = \sum_{i=1}^{n} -2x_i[y_i - (mx_i + b)] = 0$ implies $\sum_{i=1}^{n} \left(x_iy_i - mx_i^2 - bx_i\right) = 0$ or $\sum_{i=1}^{n} x_iy_i = m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i$

and $f_b = \sum_{i=1}^{n} -2[y_i - (mx_i + b)] = 0$ implies $\sum_{i=1}^{n} y_i = m\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} b = m\left(\sum_{i=1}^{n} x_i\right) + nb$. Thus we have

the two desired equations. Now $f_{mm} = \sum_{i=1}^{n} 2x_i^2$, $f_{bb} = \sum_{i=1}^{n} 2 = 2n$ and $f_{mb} = \sum_{i=1}^{n} 2x_i$. And $f_{mm}(m,b) > 0$

always and $D(m,b) = 4n\left(\sum_{i=1}^{n} x_i^2\right) - 4\left(\sum_{i=1}^{n} x_i\right)^2 = 4\left[n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2\right] > 0$ always so the

solutions of these two equations do indeed minimize $\sum_{i=1}^{n} d_i^2$.